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Free surface problems for static Coulomb-Mohr granular solids[1]

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Abstract: This paper is concerned with some plane strain and axially symmetric free surface problems which arise in the study of static granular solids that satisfy the Coulomb-Mohr yield condition. Such problems are inherently nonlinear, and hence difficult to attack analytically. Given a Coulomb friction condition holds on a solid boundary, it is shown that the angle a free surface is allowed to attach to the boundary is dependent only on the angle of wall friction, assuming the stresses are all continuous at the attachment point, and assuming also that the coefficient of cohesion is nonzero. As a model problem, the formation of stable cohesive arches in hoppers is considered. This undesirable phenomena is an obstacle to flow, and occurs when the hopper outlet is too small. Typically, engineers are concerned with predicting the critical outlet size for a given hopper and granular solid, so that for hoppers with outlets larger than this critical value, arching cannot occur. This is a topic of considerable practical interest, with most accepted engineering methods being conservative in nature. Here, the governing equations in two limiting cases (small cohesion and high angle of internal friction) are considered directly. No information on the critical outlet size is found; however solutions for the shape of the free boundary (the arch) are presented, for both plane and axially symmetric geometries.

[1]Running title: Free surface problems for granular solids

1 INTRODUCTION

Many problems in applied and industrial mathematics involve a domain which is *a priori* unknown, and has to be determined as part of the solution process. These problems are typically referred to as free or moving boundary problems, with the unknown part of the boundary being the free boundary. Traditional examples of free boundary problems arise in areas such as water waves [1], heat transfer and seepage through porous media [2], while more recently, numerous examples can be found in the study of flow through Hele-Shaw cells (see the special issue [3] and references therein), option pricing [4], tumour growth models [5, 6] and there are many more. In solid and fluid mechanics the free boundary in such problems is often referred to as a free surface, since the traction vector vanishes there, and the boundary is said to be stress-free. It is the purpose of this paper to examine some properties of free surfaces in the context of static granular materials, and in particular to use the phenomena of cohesive arching in hoppers as a model example.

Granular materials exhibit a remarkable range of complex behaviours, especially under dynamic conditions in the presence of density fluctuations. This varied behaviour has been studied extensively in the literature using a number of approaches; statistical mechanics [7, 8], molecular dynamics modelling [9], cellular automata modelling [10] and continuum mechanics [11] being some examples. On the other hand, the statics of granular materials, although still governed by nonlinear equations, is less complicated. In terms of continuum mechanics, the elimination of variation in bulk density and the absence of the inertial terms results in equations which are much more tractable. However, analytic progress in and exact solutions of static problems are still very rare, and hence when obtained make important contributions to the field (see [12], [13] and [14]-[15] for example).

The flow and storage of granular materials in hoppers and silos is a well-studied field of applied mathematics and engineering, with early contributions pioneered by Jenike [16] and Sokolovskii [12]. When an outlet at the bottom of a hopper is opened, the desirable outcome is that gravity forces the material to flow out. However, it is well known that if the size of the outlet is too small, a phenomena referred to as cohesive arching occurs, which prevents the flow of the material. In this case a stable, self-supporting obstruction (referred to here as an arch in two dimensions, and a dome in three dimensions) forms across the outlet, whose structure is strong enough to hold the net weight of the material above it. Arching is highly undesirable, as it prevents the flow of the material. In practice, an additional mechanism like vibration is needed to break an arch or dome in a hopper. The main objective is to determine, for a given hopper geometry and granular material, the critical outlet size such that for hoppers with outlet sizes above this critical value arching can never occur. For design purposes, approximate theories which overestimate this critical size have been developed by engineers (see Jenike [17, 18] and Jenike and Leser [19] for early work, or Drescher [20] for a thorough review), but due to the nonlinearities involved, exact solutions to the governing equations have never been found.

Here we consider a free boundary problem which describes the distribution of stresses near a cohesive arch that has formed across the outlet of a hopper. The equations governing a cohesive, frictional material which satisfies the Coulomb-Mohr yield condition are formulated in the following section for both plane strain and axially symmetric flow. Conditions which must be satisfied on free surfaces are noted, with special reference to limiting cases. Assuming the absence of singularities, Section 3 contains a derivation of the stress angle at the point where a free surface meets a solid boundary. It is shown that this assumption is consistent with a Coulomb friction condition acting on the boundary. Section 4 is devoted to the presentation of some limiting shapes for the unknown free surface. In particular, we consider the limits of vanishing cohesion and high angle of internal friction.

2 GOVERNING EQUATIONS

In this section we state briefly the governing equations for a static granular material subject to the Coulomb-Mohr yield condition. We also derive conditions which must be satisfied on any stress-free boundary, and discuss briefly how these conditions depend on the two material properties, namely internal friction and cohesion.

2.1 Plane strain

2.1.1 Stress equations

We consider a static granular material under the influence of gravity. Cartesian coordinates (x, y, z) are introduced so that gravity acts in the negative y direction, and plane strain conditions are so that the components of stress are independent of the z-coordinate. It follows that the equations of equilibrium are of the form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho g, \tag{1}$$

where σ_{xx}, σ_{xy} and σ_{yy} are the components of the Cauchy stress tensor in these Cartesian coordinates, g is the acceleration due to gravity and ρ the bulk density of the material, which here we assume to be a constant.

We denote the principal components of stress by σ_I, σ_{II} and σ_{III}, and order them so that $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$. Note the convention used here is that normally adopted in continuum mechanics, which is that the stress components are assumed to be positive in tension. We define the angle between the positive $x-$axis and the axis corresponding to σ_I to be ψ, and it follows that the stress components may be expressed as

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi, \tag{2}$$

where p and q are stress invariants, defined by

$$p = -\tfrac{1}{2}(\sigma_I + \sigma_{III}) = -\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}), \tag{3}$$

$$q = \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{2}\left\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right\}^{1/2}. \tag{4}$$

The stress angle ψ can be computed directly from the stress components via

$$\tan 2\psi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{5}$$

To close the system we need to prescribe a constitutive law which relates the stress invariants to the properties of the continuum. For granular materials it is commonly accepted that this can be supplied by the Coulomb-Mohr yield condition, which states that

$$q - \beta p \leq c\sqrt{1-\beta^2} \tag{6}$$

throughout the material. Here $\beta = \sin\phi$, where $0 \leq \phi \leq \pi/2$ is the angle of internal friction, and $c \geq 0$ is the coefficient of cohesion. In general, (6) holds with equality when the granular material is deforming, which is not the situation here, since we are considering static materials. However, in order to proceed, we use the rigid-plastic model, and assume that although there is no motion, the material is at the point of yielding. Therefore we have

$$q = \beta p + c\sqrt{1-\beta^2} \tag{7}$$

everywhere, unless otherwise indicated. Clearly this assumption cannot be applied to all static materials, and as such, the results derived using (7) must be treated carefully. However, in the context of the arching problem formulated in Section 4, we believe it is quite reasonable, as we explain later. We note that, in general, the cohesion c is dependent on the level of consolidation, with $c = c(\sigma_{III})$ being a common assumption. However, in the present study we assume the coefficient c is constant throughout.

At this stage we scale all lengths with respect to some appropriate length-scale l, and all stresses with respect to $\rho g l$ (including the coefficient of cohesion). For convenience we shall use the same symbols for nondimensional variables and parameters as introduced for dimensional ones; to avoid confusion, we refer to nondimensional quantities only from this point onwards.

The equilibrium equations can be rewritten in terms of the dependent variables q and ψ by substituting (2) and (7) into (1), the result being

$$\frac{\partial q}{\partial x} = \frac{\beta}{\beta^2-1}\{2q\psi_x \sin 2\psi - 2q\psi_y(\beta + \cos 2\psi) + \beta \sin 2\psi\}, \tag{8}$$

$$\frac{\partial q}{\partial y} = \frac{\beta}{\beta^2-1}\{2q\psi_x(\beta - \cos 2\psi) - 2q\psi_y \sin 2\psi + 1 - \beta\cos 2\psi\}. \tag{9}$$

These equations are standard, and hold for all values of the cohesion $c \geq 0$ and the angle of internal friction $0 \leq \phi < \pi/2$ (for $\phi = \pi/2$ we must carefully take the limit $\beta \to 1$).

2.1.2 Free surface conditions

Consider an element of surface whose unit normal is given by $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$. The traction vector on this surface is given by $\mathbf{t}^{(\mathbf{n})} = \sigma_x\mathbf{i} + \sigma_y\mathbf{j}$, where

$$\sigma_x = n_x\sigma_{xx} + n_y\sigma_{xy}, \quad \sigma_y = n_x\sigma_{xy} + n_y\sigma_{yy}.$$

On a free surface, the traction vector must vanish, so clearly

$$n_x\sigma_{xx} + n_y\sigma_{xy} = 0, \quad n_x\sigma_{xy} + n_y\sigma_{yy} = 0, \tag{10}$$

there, which implies that

$$\frac{n_x}{n_y} = -\frac{\sigma_{xy}}{\sigma_{xx}}, \quad \sigma_{xy}^2 = \sigma_{xx}\sigma_{yy},$$

on a free surface. With the use of (2) these conditions can be expressed in terms of p, q and ψ, so that, *regardless of the yield condition*, we have the two conditions

$$q = p, \quad \frac{dy}{dx} = -\frac{n_x}{n_y} = -\cot\psi, \tag{11}$$

on every free surface. Note that since $\mathbf{t}^{(\mathbf{n})} = \mathbf{0}$ on a free surface, one of the principal stresses must be zero there. This must be the greatest one (smallest in magnitude), so on a free surface $\sigma_I = 0$.

For a material which obeys the Coulomb-Mohr yield condition (7), there are three ways to interpret these free boundary conditions.

Case 1, $0 \leq \beta < 1$**,** $c > 0$. This is the most general case, and the yield condition (7) holds. On a free surface we have the two conditions

$$q = p = c\left(\frac{1+\beta}{1-\beta}\right)^{1/2}, \quad \frac{dy}{dx} = -\cot\psi. \tag{12}$$

Case 2, $0 \leq \beta < 1$**,** $c = 0$. Here the yield condition states that $q = \beta p$ everywhere. Therefore the only way (11a) can be satisfied is if $q = p = 0$ on the free surface. So in this case *all* the stress components σ_{xx}, σ_{xy}, and $\sigma_{yy} = 0$ vanish on a free surface and hence (10) reveals nothing about n_x and n_y. That is, *any* surface for which $q = p = 0$ is a free surface.

Case 3, $\beta = 1$**,** $c \geq 0$. Here the yield condition states that $q = p$ everywhere. This implies that $\sigma_I = 0$ everywhere, and *any* surface for which

$$\frac{dy}{dx} = -\cot\psi$$

is a stress-free surface. That is to say that in the limiting case $\beta = 1$, the equations are such that all problems become natural candidates for free boundary problems. We investigate this phenomena further in Section 4.4, when we consider arching in highly frictional materials.

We note in passing that from (12) we find that for cohesive materials $\sigma_{III} = -f_c$ on a free surface, where

$$f_c = 2c\left(\frac{1+\beta}{1-\beta}\right)^{1/2},$$

which is often referred to as the unconfined yield strength of the material.

2.2 Axially symmetric strain

2.2.1 Stress equations

Cylindrical polar coordinates (r, φ, z) are introduced so that gravity acts in the negative z direction. If the state of stress is axially symmetric, then the (nondimensional) equations of equilibrium are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \quad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 1, \tag{13}$$

where σ_{rr}, σ_{rz} and σ_{zz} are the components of the Cauchy stress tensor in these cylindrical polar coordinates. We define ψ to be the angle between the maximum principal stress axis and the r-direction, and it follows that

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{zz} = -p - q\cos 2\psi, \quad \sigma_{rz} = q\sin 2\psi, \tag{14}$$

where p and q are the positive quantities, defined by

$$p = -\tfrac{1}{2}(\sigma_{rr} + \sigma_{zz}), \quad q = \tfrac{1}{2}\left\{(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right\}^{1/2}. \tag{15}$$

The stress angle ψ is related to the Cauchy stress components by

$$\tan 2\psi = \frac{2\sigma_{rz}}{\sigma_{rr} - \sigma_{zz}}. \tag{16}$$

As with the plane-strain case considered above, it is assumed the Coulomb-Mohr yield condition holds, even though the material is not deforming. Again, we justify this assumption in the arching problem considered later by arguing the material is at the point of yielding (see Section 4 for details). There are several axially symmetric regimes which are consistent with this yield condition, and these are listed in Spencer [11] and Cox et al. [21]. The appropriate case for the arching problem considered in Section 4 is the Haar-von Karmann regime for which the hoop stress, which is a principal stress, be equal to the maximum principal stress. Thus

$$\sigma_I = \sigma_{\varphi\varphi} = \sigma_{II} \geq \sigma_{III},$$

and it follows that the hoop stress is given by

$$\sigma_{\varphi\varphi} = -p + q. \tag{17}$$

With this regime the yield condition again takes the form (7), where this time p and q are given by (15). Note that with these assumptions, the invariants p and q can also be expressed as

$$p = -\tfrac{1}{2}(\sigma_I + \sigma_{III}), \quad q = \tfrac{1}{2}(\sigma_I - \sigma_{III}),$$

as in the two-dimensional case considered previously, and in fact all the equations and results derived for axially symmetric strain are analogous to the plane strain case. As before, we use (7) and (14) to write the equilibrium equations in terms of q and ψ and the result is

$$\frac{\partial q}{\partial r} = \frac{\beta}{\beta^2 - 1}\left\{2q\psi_r \sin 2\psi - 2q\psi_z(\beta + \cos 2\psi) + \beta \sin 2\psi + \frac{q(\beta - 1)(\cos 2\psi - 1)}{r}\right\}, \tag{18}$$

$$\frac{\partial q}{\partial z} = \frac{\beta}{\beta^2 - 1}\left\{2q\psi_r(\beta - \cos 2\psi) - 2q\psi_z \sin 2\psi + 1 - \beta \cos 2\psi + \frac{q(\beta - 1)\sin 2\psi}{r}\right\}. \tag{19}$$

2.2.2 Free surface conditions

Consider an axially symmetric surface whose normal is given by $\mathbf{n} = n_r\mathbf{e}_r + n_\varphi\mathbf{e}_\varphi + n_z\mathbf{e}_z$. The traction vector on this surface can be written $\mathbf{t}^{(\mathbf{n})} = \sigma_r\mathbf{e}_r + \sigma_\varphi\mathbf{e}_\varphi + \sigma_z\mathbf{e}_z$, where

$$\sigma_r = n_r\sigma_{rr} + n_z\sigma_{rz}, \quad \sigma_\varphi = n_\varphi\sigma_{\varphi\varphi}, \quad \sigma_z = n_r\sigma_{rz} + n_z\sigma_{zz},$$

so if the surface is a free surface, then we must have

$$\frac{n_r}{n_z} = -\frac{\sigma_{rz}}{\sigma_{rr}}, \quad \sigma_{rz}^2 = \sigma_{rr}\sigma_{zz}$$

there. In terms of p, q and ψ, these two conditions are

$$q = p, \quad \frac{dz}{dr} = -\frac{n_r}{n_z} = -\cot\psi, \tag{20}$$

where again we note that these conditions are independent of the yield condition.

Now if we are to assume the yield condition (7) applies, then there are three possibilities, which are essentially the same as the plane strain case.

Case 1, $0 \le \beta < 1$, $c > 0$. Here on the free surface we have

$$q = p = c\left(\frac{1+\beta}{1-\beta}\right)^{1/2}, \quad \frac{dz}{dr} = -\cot\psi. \tag{21}$$

Case 2, $0 \le \beta < 1$, $c = 0$. Here $q = p = 0$ on a free surface.

Case 3, $\beta = 1$, $c \ge 0$. Here $q = p$ everywhere, and *any* surface for which

$$\frac{dz}{dr} = -\cot\psi$$

is a stress-free surface.

3 INTERSECTION OF FREE SURFACES WITH SOLID BOUNDARIES

In many instances granular materials are stored in solid containers, and often traction-free surfaces intersect the container walls. Naturally this will occur at the upper surface of the material, but in some circumstances a material may be bounded below by a traction-free surface (for example, when arches or domes form in hoppers). Another typical intersection point where free surfaces meet solid boundaries occurs when material is stored in granular heaps. Here the solid boundary may be simply the ground, or some other horizontal surface.

In this section we consider the Coulomb friction condition at these solid boundaries, and investigate stress fields in the neighbourhood of a point where a free surface intersects such a boundary. Provided there is no singularity of stress, we are able to derive the stress angle ψ at the intersection point, and hence determine the angle the free surface must make with the solid boundary.

3.1 Coulomb friction condition

3.1.1 Plane strain

Consider a surface element in the (x, y)-plane whose unit normal is given by $\mathbf{n} = \cos\chi\mathbf{i} + \sin\chi\mathbf{j}$, where χ is the angle between $\mathbf{n}$ and the x-axis. With the use of (2), we find the normal component of compressive traction along the surface element to be

$$\sigma_n = p - q\cos 2(\chi - \psi), \tag{22}$$

and the magnitude of the shear component of traction to be

$$\tau_n = q|\sin 2(\chi - \psi)|. \tag{23}$$

Now suppose the surface element is actually a solid boundary, and that the normal vector points towards the granular material. The Coulomb friction condition states that

$$\tau_n \leq \sigma_n \tan\mu \tag{24}$$

along the boundary, where μ is the angle of wall friction. For definiteness we assume that $0 \leq \chi \leq 2\pi$, and for each fixed value of χ we allow possible values of ψ to lie in the range $\chi - \pi \leq \psi \leq \chi$. The inequality (24) can now be rewritten as

$$q\sin(|2(\chi - \psi) - \pi| - \mu) \leq p\sin\mu \tag{25}$$

on the solid boundary.

In general, the conditions (24)-(25) will hold with inequality under static circumstances, and with equality when the material is slipping across the boundary. However, the material may be at the point of slipping, so that both (24) and (25) hold with equality even though there is no motion. We note that if the material is at the point of slipping across the solid boundary, then (25) has no meaning if $p\sin\mu > q$. In this case the solid boundary is perfectly rough, and the material is at the verge of slipping on itself. The appropriate condition here is that

$$|2(\chi - \psi) - \pi| = \frac{\pi}{2} + \arcsin\left(\frac{q}{p}\right) \tag{26}$$

on the solid boundary.

3.1.2 Axially symmetric strain

Consider an axially symmetric solid boundary, whose unit normal is give by $\mathbf{n} = \cos\chi\,\mathbf{e}_r + \sin\chi\,\mathbf{e}_z$, where χ is the angle between $\mathbf{n}$ and the r-axis. With the use of (14), we find the normal component of compressive traction along the boundary and the magnitude of the shear component of traction to be (22) and (23) respectively. It follows that the friction condition which applies on an axially symmetric boundary is (essentially) the same as that presented in Section 3.1.1. That is, we have (25) on a solid boundary, provided $p\sin\mu \geq q$; otherwise the boundary is consider to be perfectly rough, and the condition (26) holds (here p and q are defined in (15)).

3.2 Angle between free surface and solid boundary

Suppose in two dimensions there is a free surface which intersects the solid boundary at the point P, as depicted in Figure 1(a). In the immediate neighbourhood of P, there is a wedge enclosed by an angle Ω, say. Let the stresses at an internal point of the wedge have components σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$, where r and θ are variables defined in Figure 1(b). (These variables are not to be confused with those used later in Section 4.) At the point P, these stresses are functions of θ only; at $\theta = 0$ we must have $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$.

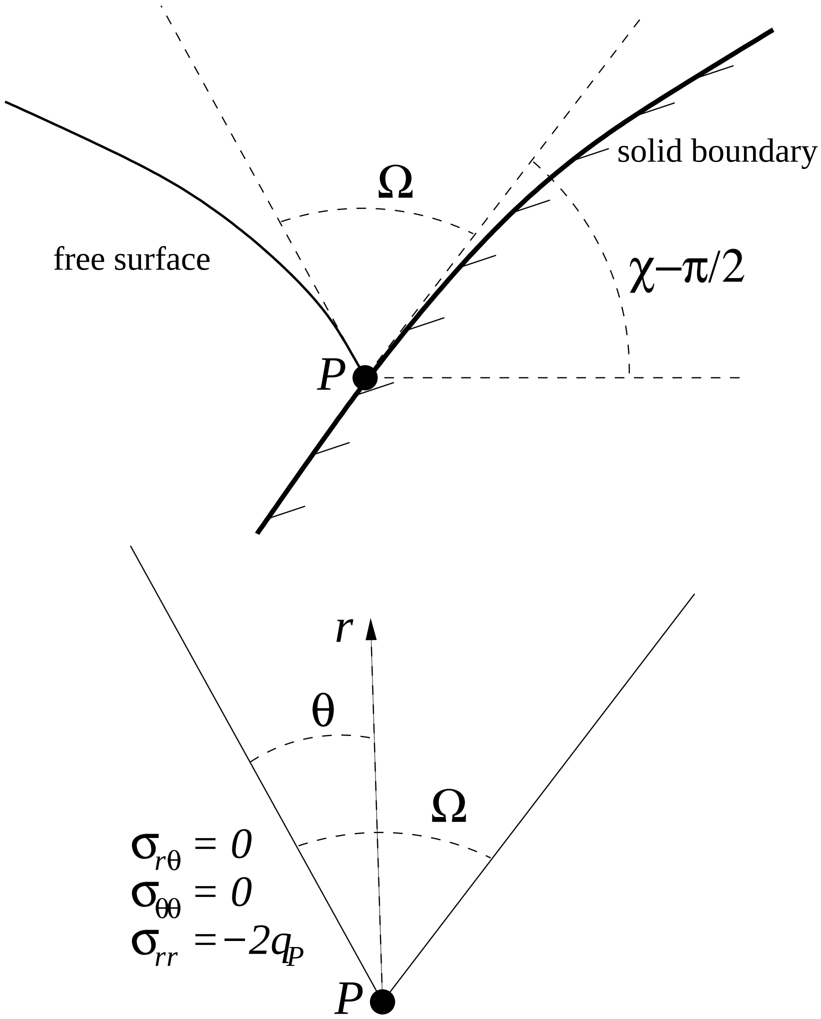

Fig. 1: A definition sketch of an intersection between a free surface and a solid boundary.

As discussed by R. Hill [22], an arbitrary loading of a wedge such as that shown in Figure 1(b) will lead to a singularity of stress. However, if we assume that all quantities are continuous at P, the equilibrium equations, given here by

$$\frac{d\sigma_{r\theta}}{d\theta} + \sigma_{rr} - \sigma_{\theta\theta} = 0, \quad \frac{d\sigma_{\theta\theta}}{d\theta} + 2\sigma_{r\theta} = 0,$$

are satisfied by the functions

$$\sigma_{rr} = -q_P(1 + \cos 2\theta), \quad \sigma_{r\theta} = q_P \sin 2\theta, \quad \sigma_{\theta\theta} = -q_P(1 - \cos 2\theta),$$

where q_P is the value of q at P. Thus, under this assumption, a necessary condition on the wall is that

$$\sigma_{r\theta}(\Omega) = q_P \sin 2\Omega, \quad \sigma_{\theta\theta}(\Omega) = -q_P(1 - \cos 2\Omega),$$

which implies that

$$\left|\frac{\sigma_{r\theta}}{\sigma_{\theta\theta}}\right| = \left|\tan(\tfrac{1}{2}\pi - \Omega)\right| \quad \text{at} \quad \theta = \Omega. \tag{27}$$

We can conclude from this argument that the Coulomb friction condition (24) applied with strict equality is consistent with continuity of stress at P provided the angle between the free surface and the wall is

$$\Omega = \frac{\pi}{2} \pm \mu, \tag{28}$$

and the stress angle at P is given by

$$\psi = \chi \pm \mu - \frac{\pi}{2}. \tag{29}$$

Otherwise, there must be a singularity of stress, as described by R. Hill [22].

No reference has been made in this section to the yield condition (7). With $0 \leq \beta < 1$, $c > 0$, the results are valid for a Coulomb Mohr material provided the value of q at the point P is given by

$$q_P = c\left(\frac{1+\beta}{1-\beta}\right)^{1/2}$$

(see (12)). However, the above analysis does not hold for $0 \leq \beta < 1$, $c = 0$, since in this case $q = p = 0$ on a free surface (see Case 2 in Section 2.1.2) and (27) makes no sense. No information about the stress angle at the intersection point may be gained here. For the limiting situation $\beta = 1$, $c \geq 0$, we have $q = p$ everywhere (see Case 3 in Section 2.1.2). Therefore the result (29) derived above for ψ at the intersection point P holds for every point on the solid boundary.

For the axially symmetric case, we suppose the free surface intersects the solid boundary at a circle, and denote an arbitrary point on this circle by P. The above argument in two dimensions is also valid here, since by (17) and (21) the hoop stress vanishes on the free surface, and the geometry is two-dimensional in the neighbourhood of P. That is, if we assume the stresses are all continuous at P, the Coulomb friction condition (with an angle of wall friction μ) applies on the solid boundary, and that the cohesion is nonzero, then the free surface must necessarily intersect the boundary with an angle enclosed by (28) and stress angle given by (29).

4 ARCHING AND DOMING IN HOPPERS

4.1 Formulation for plane hoppers

As previously discussed, we wish to consider the free boundary problem which arises when discharge out of a hopper is halted by the formation of a stable, cohesive arch. We denote the outlet diameter at the bottom of the hopper by B, and assume there exists a critical value B_c such that, for a given hopper and material, arches will always form for $B \leq B_c$, and never form for $B > B_c$. In reality, the situation is not quite so straightforward, and other factors such as material loading history, the degree of compaction and inhomogeneity all have to be considered, but here we ignore such complications.

We suppose now that $B = B_c$, so that the stable arch is on the verge of being broken. It is therefore reasonable to assume that the material everywhere in the neighbourhood of the free surface is at yield, given that any widening of the outlet will lead to discharge. Accordingly, the Coulomb-Mohr condition (7) is appropriate. For plane strain, the equilibrium equations (1) must also hold, so the governing equations for stresses in a plane hopper are given by (8)-(9). It remains to consider the boundary conditions.

It will prove convenient to use polar coordinates, as indicated in Figure 2, and define the hopper walls by $\theta = \gamma$ and $\theta = \pi - \gamma$. Note the variables r and θ used in this section are not the same as those used in Section 3.2. We define the shape of the free surface by $r = \eta(\theta)$. On this surface we have either of the three conditions in Section 2.1.2 (Cases 1, 2 or 3) depending on the values of the material parameters β and c. There is also the symmetry condition

$$\psi = \frac{\pi}{2} \quad \text{on} \quad \theta = \frac{\pi}{2}, \tag{30}$$

so we need only consider $\gamma \leq \theta \leq \pi/2$.

We assume that the material is at the point of slipping down the hopper wall, so that

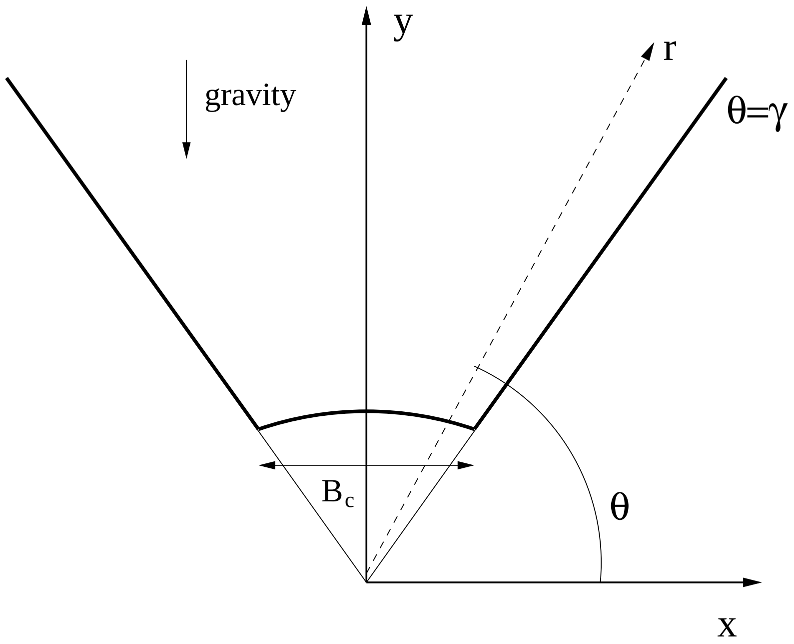

Fig. 2: Schematic for plane hopper

from (7) and (25) we have

$$\psi = \gamma - \frac{1}{2}\mu - \frac{1}{2}\arcsin\left[\frac{\sin\mu}{\beta}\left(1 - \frac{c}{q}\sqrt{1-\beta^2}\right)\right] \quad \text{on} \quad \theta = \gamma, \tag{31}$$

provided

$$\sin\mu \leq \beta\left(1 - \frac{c}{q}\sqrt{1-\beta^2}\right)^{-1}.$$

Otherwise, the material slips on itself, and from (26) we have

$$\psi = \gamma - \frac{\pi}{4} - \frac{1}{2}\arcsin\left[\beta\left(1 - \frac{c}{q}\sqrt{1-\beta^2}\right)\right] \quad \text{on} \quad \theta = \gamma. \tag{32}$$

We note in passing that, unless the angle of wall friction $\mu = 0$, the possibility of the arches being arcs of circles can immediately be ruled out, since $\psi \neq \gamma$ at the point where the arch intersects the hopper.

Finally, we impose the conditions

$$q \sim q_\infty(r,\theta), \quad \psi \sim \psi_\infty(r,\theta) \quad \text{as} \quad r \to \infty, \tag{33}$$

where q_∞ and ψ_∞ are solutions to (8)-(9) with the above boundary conditions in an infinite half-hopper $0 \leq r < \infty$, $\gamma \leq \theta \leq \pi/2$. That is to say we are only interested in the stresses that form in the vicinity of the arch. The current problem could be considered an inner problem, which could in principle be matched with an outer problem that addresses the appropriate considerations at the top traction-free surface. However, we do not consider such an outer problem here. Given the solution for the location of the free surface, the critical outlet width can be recovered via $B_c = 2\eta(\gamma)\sin\gamma$.

4.2 Formulation for conical hoppers

As with the two-dimensional case just described, we assume that in our problem the outlet diameter $B = B_c$, where B_c is the critical value (stable, cohesive domes will always form for $B \leq B_c$, and are unable to form for $B > B_c$). It follows that the material in the neighbourhood of the outlet is at the point of yielding, and thus the yield condition (7) applies, and that the required governing equations are (18)-(19).

We now introduce the coordinates

$$R = \sqrt{r^2 + z^2}, \quad \theta = \arctan(z/r),$$

which are illustrated in Figure 3, and let the (axially symmetric) hopper wall be defined by $\theta = \gamma$. On the hopper wall we assume the material is at the point of slipping, so that one of the conditions (31) or (32) hold, depending on the value of the angle of wall friction μ, as described in Section 4.1. Again, as with the plane hopper, there is the symmetry condition (30), and we need only consider stresses in the region $\gamma \leq \theta \leq \pi/2$.

In the present axially symmetric case, the surface of the dome is denoted by $R = \zeta(\theta)$. Since this is a stress-free surface, one of the three conditions in Section 2.2.2 applies (Case 1, 2 or 3), depending on the value of β and c.

Finally, there are the conditions

$$q \sim q_\infty(R,\theta), \quad \psi \sim \psi_\infty(R,\theta) \quad \text{as} \quad R \to \infty. \tag{34}$$

Here q_∞ and ψ_∞ are solutions to (18)-(19) with the above boundary conditions in the infinite hopper $0 \leq R < \infty$, $\gamma \leq \theta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$. As mentioned in Section 4.1, these conditions represent a matching with an outer problem which we do not consider. The

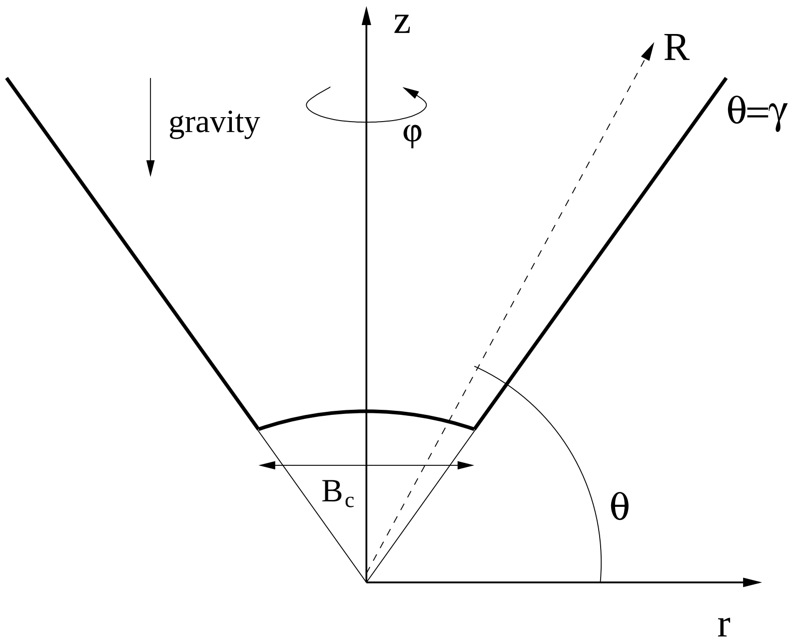

Fig. 3: Schematic for conical hopper

result is that we only consider behaviour in the neighbourhood of the dome, and hence cannot describe stresses near a traction-free surface which may bound the material above.

4.3 Granular materials with small cohesion

4.3.1 Plane hopper

By referring to Case 2 in Section 2.1.2, we find that when the cohesion $c = 0$, the condition $q = 0$ applies on the arching free surface. From (9), it follows that

$$\frac{\partial q}{\partial y} = -\frac{\beta(1-\beta\cos 2\psi)}{1-\beta^2} < 0$$

there, implying we must have $q < 0$ in the material. This is impossible, since by definition $q \geq 0$, so we arrive at the well-known conclusion that stable arches cannot form when $c = 0$. Furthermore, we must have $B_c \to 0$ as $c \to 0$.

This reasoning suggests that for $0 < c \ll 1$, we should write

$$q \sim q_0(r,\theta) + O(c), \quad \psi \sim \psi_0(r,\theta) + O(c), \quad \eta \sim c\,\eta_0(\theta) + O(c^2) \quad \text{as} \quad c \to 0. \qquad (35)$$

Substituting these expressions into the governing equations (8)-(9) gives the same equations to leading order, with q and ψ replaced with q_0 and ψ_0 respectively.

The condition (31) on the hopper wall becomes, to leading order,

$$\psi_0 = \gamma - \frac{1}{2}\mu - \frac{1}{2}\arcsin\left(\frac{\sin\mu}{\sin\phi}\right) \quad \text{on} \quad \theta = \gamma, \tag{36}$$

where for convenience we have replaced β with $\sin\phi$. This condition is meaningful provided $\mu \leq \phi$. For $\mu > \phi$, the hopper wall is perfectly rough, and from (32) we have

$$\psi_0 = \gamma - \frac{\pi}{4} - \frac{1}{2}\phi \quad \text{on} \quad \theta = \gamma. \tag{37}$$

The symmetry condition (30) is simply

$$\psi_0 = \frac{\pi}{2}, \quad \text{on} \quad \theta = \frac{\pi}{2}, \tag{38}$$

and we are left to deal with the free boundary conditions. The relevant conditions are (12), since even though the cohesion is small, we still have $c > 0$. From (12a) we find

$$q_0 = 0 \quad \text{at} \quad r = 0; \tag{39}$$

the consideration of (12b) is delayed until later.

Now the solution to (8)-(9) with q and ψ replaced with q_0 and ψ_0 subject to the boundary conditions (36)-(39) is the so-called radial solution

$$q_0 = rF(\theta), \quad \psi_0 = \bar{\psi}_0(\theta),$$

where F and $\bar{\psi}_0$ satisfy the coupled ordinary differential equations

$$\frac{dF}{d\theta} = \frac{F\sin 2(\bar{\psi}_0 - \theta) - \beta\cos(2\bar{\psi}_0 - \theta)}{\beta + \cos 2(\bar{\psi}_0 - \theta)}$$

$$\frac{d\bar{\psi}_0}{d\theta} = \frac{(1-\beta^2)F + \beta^2\sin(2\bar{\psi}_0 - \theta) + \beta\sin\theta}{2\beta F(\beta + \cos 2(\bar{\psi}_0 - \theta))}.$$

This solution has received much attention in the literature, beginning with both Sokolovskii [12] and Jenike [16]. These ordinary differential equations can easily be solved numerically; the reader is referred to Hill and Cox [23] for details, noting that we are using a slightly different coordinate system than previous authors.

So for small values of the cohesion c, the critical outlet width $B_c = O(c)$, and the limiting arch is located close to the hopper apex. In this limit the leading order solution for the stresses is just the well-known radial solution. Information about the shape of the limiting arch can be found from (12b), which implies that the slope of the leading order free boundary description is given by

$$\begin{aligned}\frac{dy}{dx} &= -\cot\psi_0(r,\theta) \quad \text{at} \quad r = 0 \\ &= -\cot\bar{\psi}_0(\theta)\end{aligned}$$

Unfortunately this limiting process only reveals the slope of the free surface as a function of angle θ, and information about the critical outlet width B_c is lost. However, it is of interest to view the shape of the free surface, and an example is presented in Figure 4. Here we have made the scaling $\eta_0 = \lambda\bar{\eta}(\theta)$, and fixed $\bar{\eta}(\pi/2) = 1$. Note that only the right half of the arch has been shown in this figure.

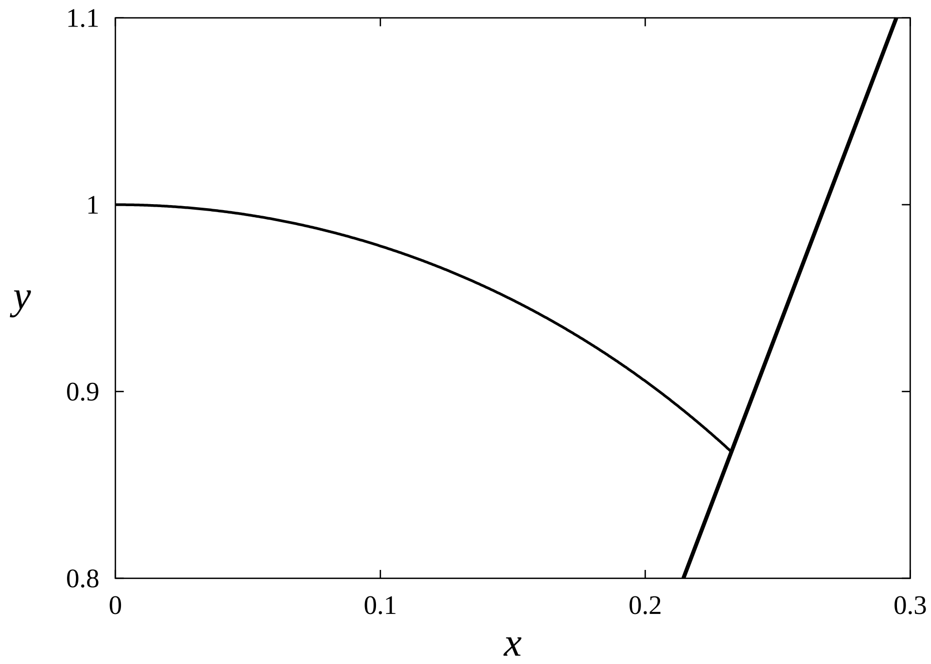

Fig. 4: Typical free surface profile $r = \eta_0(\theta)$ for $0 < c \ll 1$. The plot shown is for $\phi = \pi/4$, $\gamma = 5\pi/12$, $\mu = \pi/6$ and $\lambda = 1$.

It should be mentioned that the angle the leading order free surface shape makes with the hopper wall is different from what the exact solution would yield, as predicted by in Section 3.2; however it is expected that higher order terms in the expansion (35) would correct this difference.

4.3.2 Conical hopper

Using an analogous argument to that just presented in Section 4.3.1, we can easily show that stable, cohesive domes cannot form for purely frictional ($c = 0$) materials, and that $B_c \to 0$ as $c \to 0$. We are lead to seek asymptotic solutions by writing

$$q \sim q_0(R, \theta) + O(c), \quad \psi \sim \psi_0(R, \theta) + O(c), \quad \zeta \sim c\,\zeta_0(\theta) + O(c^2) \quad \text{as} \quad c \to 0.$$

Using these expansions, we find that, to leading order, the governing equations are the same. That is, we have to solve (18)-(19) with q and ψ replaced with q_0 and ψ_0 respectively. It turns out that the boundary conditions (36)-(39), derived for the plane strain case, are also applicable in the axially symmetric case considered presently.

The solution to (18)-(19) with (q, ψ) replaced with (q_0, ψ_0) which satisfies the boundary conditions (36)-(39) is the radial solution

$$q_0 = RG(\theta), \quad \psi_0 = \bar{\psi}_0(\theta),$$

with G and $\bar{\psi}_0$ solutions to the ordinary differential equations

$$\frac{dG}{d\theta} = \frac{2\cos(\bar{\psi}_0 - \theta)(\sin(\bar{\psi}_0 - \theta) + \beta \sec\theta \sin\bar{\psi}_0)G - \beta\cos(2\bar{\psi}_0 - \theta)}{\beta + \cos 2(\bar{\psi}_0 - \theta)}$$

$$\frac{d\bar{\psi}_0}{d\theta} = \frac{(1-\beta)(1 + 2\beta - \beta\sec\theta\cos(2\bar{\psi}_0 - \theta))G + \beta\sin\theta + \beta^2\sin(2\bar{\psi}_0 - \theta)}{2\beta G(\beta + \cos 2(\bar{\psi}_0 - \theta))}$$

As with the two-dimensional counterpart, this solution has been studied widely in the literature, beginning with Jenike [16].

We are left to conclude that for $0 < c \ll 1$, the leading order stress field that describes material in a conical hopper bounded below by a stable, cohesive dome is the well-known radial solution. In addition, the slope of the free surface can be found from (21), and is given by

$$\frac{dz}{dr} = -\cot\psi_0(0, \theta) = -\cot\bar{\psi}_0(\theta).$$

As with the two-dimensional case, we comment that this limiting process does not reveal the location of the limiting dome; only the shape can be recovered. A typical free surface profile is shown in Figure 5. Here we have made the rescaling $\zeta_0 = \lambda\bar{\zeta}$, and fixed $\bar{\zeta}(\pi/2) = 1$.

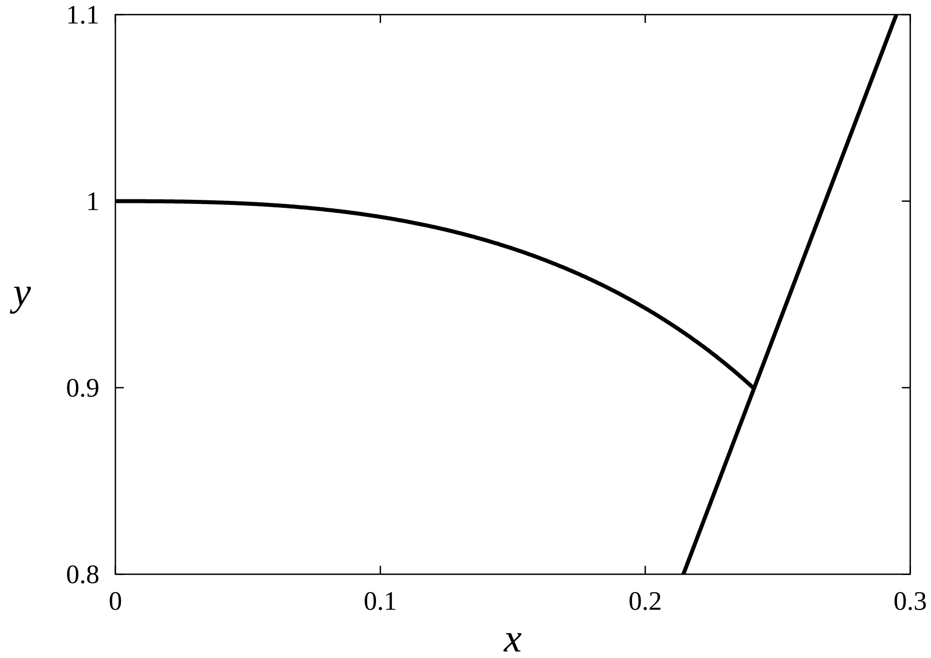

Fig. 5: Typical free surface profile $R = \zeta_0(\theta)$ for $0 < c \ll 1$. The plot shown is for $\phi = \pi/4$, $\gamma = 5\pi/12$, $\mu = \pi/6$ and $\lambda = 1$.

4.4 Highly frictional granular materials

Much progress has been made recently in the determination of exact solutions to the nonlinear governing equations (8)-(9) for the limiting case $\beta = 1$ (see [14], [15], [23], for example). The solutions presented are bonafide exact solutions of the field equations and represent ideal limiting behaviour of real physical materials.

There are compelling reasons for studying highly frictional materials (those for which $\beta = 1$). The equations for $\beta = 1$ can be used to describe the leading order term in a regular perturbation with $1 - \beta \ll 1$. Even for moderately large values of the angle of internal friction ϕ, the quantity $1-\beta$ is small, and so the leading order approximation is valid. As an example, the reader is referred to Hill and Cox [23], who consider the flow of granular materials through a hopper. Here, figures show the (exact) solution for $\phi = \pi/2$ ($\beta = 1$) is quantitatively similar to the (numerical) solution for $\phi = \pi/3$ ($\beta = 0.87$). In this case it is easily seen that the solution for $\beta = 1$ provides a limiting bound, or envelope, for physically meaningful materials. Granular materials exhibiting high angles of internal friction have been reported in Sture [24], for example.

As with all applied mathematical modelling, there is a trade off between physical applicability and mathematical tractability. The equations for highly frictional materials admit exact solutions which do not exist for general values of the angle of internal friction. These exact solutions can be used as a check for numerical schemes that are written primarily for $\beta \neq 1$. Furthermore, such exact solutions are of an interest in their own right, given that any analytic progress to highly nonlinear fields such as granular materials is invaluable.

In the context of the present study we are particularly interested in the limit $\beta = 1$, for as well as providing possible exact solutions, the theory for highly frictional materials provides problems which are natural candidates for free surface problems. This is because one of the two free surface conditions is identically satisfied, and hence embedded in any such solution are possible free surfaces. The result is that we are able to find exact solutions to free surface problems, which based on other areas in mechanics are known to be very rare.

4.4.1 Plane hopper

The governing equations (8) and (9) can be arranged as

$$(\beta-1)(q_x\cos\psi + q_y\sin\psi) = \beta\sin\psi + 2\beta q(\psi_x\sin\psi - \psi_y\cos\psi),$$

$$(\beta+1)(q_x\sin\psi - q_y\cos\psi) = \beta\cos\psi - 2\beta q(\psi_x\cos\psi + \psi_y\sin\psi),$$

so that, in the special case $\beta = 1$, we find

$$q = \frac{-1}{2(\psi_x - \psi_y\cot\psi)}. \tag{40}$$

The stress angle ψ can then be shown to satisfy the nonlinear partial differential equation

$$h_{xx} - 2hh_{xy} + h^2 h_{yy} = 0, \tag{41}$$

where $h(x, y) = \cot\psi$. There is an exact solution to (41) which is given parametrically by

$$\cot\theta = \frac{b}{2s^{-1/2}e^{s/2} - I(s)}, \quad \cot\psi = \frac{1}{b}I(s), \tag{42}$$

where

$$I(s) = \int_a^s \frac{e^{\omega/2}}{\omega^{1/2}}\, d\omega \tag{43}$$

and a and b are constants of integration. The corresponding solution for q, computed from (40), is given by

$$q = \frac{r}{4}\frac{s^{-1/2}e^{-s/2}(I(s)^2 + b^2)}{[(2s^{-1/2}e^{s/2} - I(s))^2 + b^2]^{1/2}}.$$

This solution is due to Hill and Cox [23].

From (31), the condition on the hopper wall becomes

$$\psi = \gamma - \mu \quad \text{on} \quad \theta = \gamma \tag{44}$$

for $\beta = 1$. Note how this agrees with (29). The boundary conditions (30) and (44) can be satisfied by (42)-(43) if we choose $a = 0$ and

$$b = (2s_0^{-1/2}e^{s_0/2} - I(s_0))\cot\gamma,$$

with s_0 the solution to

$$(2s_0^{-1/2}e^{s_0/2} - I(s_0))\cot\gamma\cot(\gamma - \mu) - I(s_0) = 0.$$

This exact solution for $\beta = 1$ can be identified with q_∞ and ψ_∞ in (33). A consequence of the yield condition (7) is that for $\beta = 1$ the coefficient of cohesion c does not appear in the boundary conditions (30) and (44), and hence (since it also does not appear in either (8) or (9)) plays no part in the solution.

Now this is Case 3 in Section 2.1.2, where *any* surface for which $dy/dx = -\cot\psi$ is a free surface. So there is an infinite family of these stress-free curves, whose slope depends on θ only. Furthermore, each point in space belongs to only one of these curves, and given a particular point, the corresponding stress-free surface can easily be computed numerically. We can conclude that for the limiting case $\beta = 1$, the free boundary problem is not unique, and that for given values of hopper wall angle γ and angle of wall friction μ, we cannot compute the critical outlet width B_c. However, we have constructed exact solutions to a free boundary boundary, which is an exception, and hence worthy of presentation.

An example of the stress-free surfaces predicted by this exact solution is presented in Figure 6. For definiteness we have chosen to force $\eta(\pi/2) = 1$, but as already noted, this value of η cannot be determined by our solution.

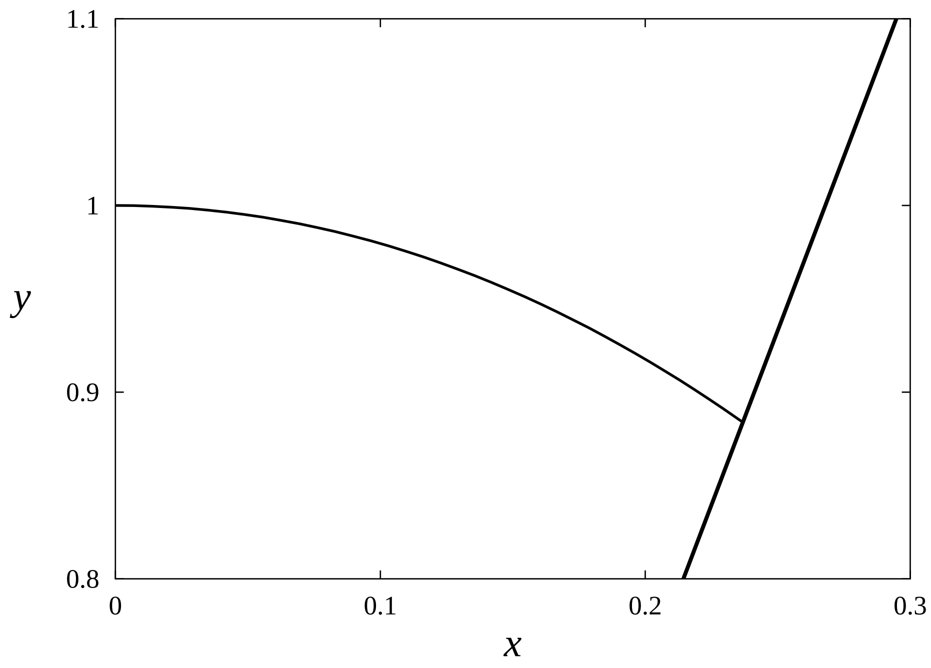

Fig. 6: Typical free surface profile $r = \eta(\theta)$ for $\beta = 1$. The plot shown is for $\gamma = 5\pi/12$, $\mu = \pi/6$.

4.4.2 Conical hopper

For the limiting case of $\beta = 1$ we can solve (18) and (19) directly for q to give

$$q = \frac{-1}{2(\psi_r - \psi_z \cot \psi)},$$

with the resulting nonlinear partial differential equation for ψ given by

$$h_{rr} - 2hh_{rz} + h^2 h_{zz} - \frac{1}{r}(h_r - hh_z) = 0,$$

where $h(r,z) = \cot\psi$. There is an exact solution in which the stress angle ψ depends on the angle θ only, and it is given by

$$\cot\theta = \frac{b}{3s^{-1/2}e^{s/3} - I(s)}, \quad \cot\psi = \frac{1}{b}I(s), \tag{45}$$

$$q = \frac{R}{6}\frac{s^{-2/3}e^{-s/3}(I(s)^2 + b^2)}{[(3e^{s/3}s^{-1/3} - I(s))^2 + b^2]^{1/2}}, \tag{46}$$

with

$$I(s) = \int_a^s \frac{e^{\omega/3}}{\omega^{1/3}}\, d\omega, \tag{47}$$

remembering that $R = (r^2 + z^2)^{1/2}$ and $\theta = \arctan(r/z)$. This solution was first presented by Cox and Hill [15].

For a granular material at yield in a hopper, the symmetry condition (30) and the hopper wall condition (44) can be satisfied by the exact solution (45)-(47) if we choose

$$a = 0 \quad \text{and} \quad b = (3s_0^{-1/3}e^{s_0/3} - I(s_0))\cot\gamma, \tag{48}$$

with s_0 the root of

$$(3s_0^{-1/3}e^{s_0/3} - I(s_0))\cot\gamma\cot(\gamma - \mu) - I(s_0) = 0. \tag{49}$$

This solution therefore describes the stresses that form when material is at yield in the infinite hopper $0 \le R < \infty$, $\gamma \le \theta \le \pi/2$, $0 \le \varphi \le 2\pi$. But here we are considering the limiting situation $\beta = 1$, which is Case 3 in Section 2.2.2. Hence we know that along any surface for which $dz/dr = -\cot\psi$, the traction vector vanishes, and the surface is stress-free. We can therefore identify one of these stress-free surfaces as a possible self-supporting dome, in which case (45)-(49) is also an exact solution to the free boundary problem (18)-(19), (30)-(31), (34) with (20) on $R = \zeta(\theta)$

Figure 7 contains an example of the stress-free surfaces predicted by this exact solution. As with the plane strain case, we have chosen to force $\zeta(\pi/2) = 1$, but note that stress-free surfaces exist for other values of $\zeta(\pi/2)$.

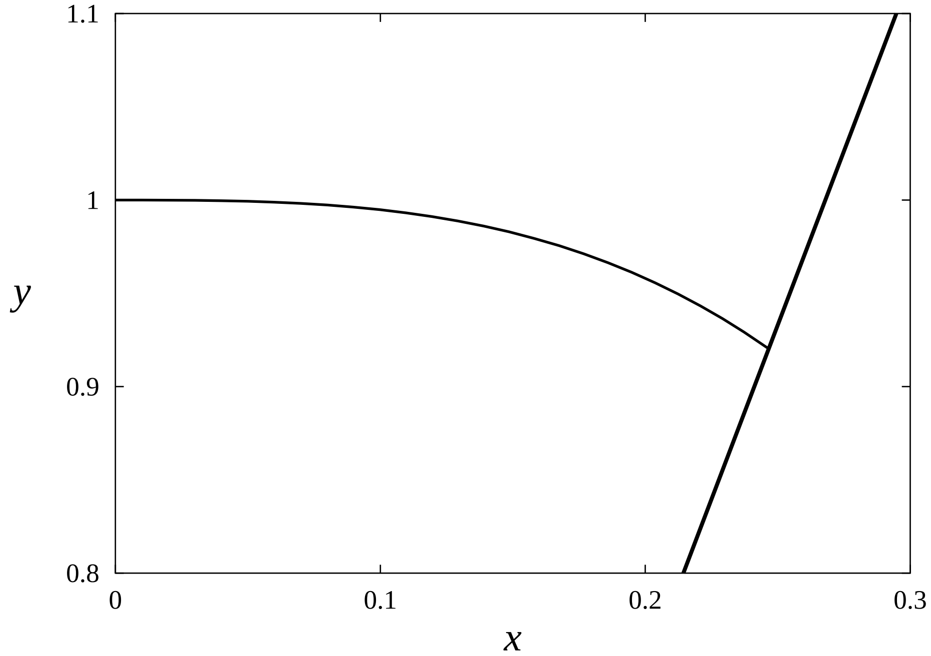

Fig. 7: Typical free surface profile $R = \zeta(\theta)$ for $\beta = 1$. The plot shown is for $\gamma = 5\pi/12$, $\mu = \pi/6$.

5 DISCUSSION

For a static granular material, it is easy to derive the two conditions (11) or (20) which must be satisfied along a stress-free surface. In general, these conditions are independent of the yield condition. In terms of materials which satisfy the Coulomb-Mohr yield condition (7), special care needs to be taken in the limiting cases of cohesion $c = 0$ and internal friction $\beta = 1$. For cohesionless materials (Case 2 in Sections 2.1.2 and 2.2.2), every component of the stress tensor must vanish on a stress-free surface, and hence the shape of these stress-free surfaces is independent of the stress angle ψ. For highly frictional materials (Case 3 in Sections 2.1.2 and 2.2.2), the condition $q = p$ is identically satisfied. Therefore, embedded in every solution with $\beta = 1$ are possible stress-free surfaces.

By considering the stresses in the neighbourhood of an intersection between a free surface and a solid boundary, we show that a Coulomb friction condition on the solid boundary is consistent with the assumption that there is no singularity in the stress field.

Furthermore, the angle of attachment is found to depend on the angle of wall friction only. This analysis holds for granular solids regardless of the yield condition, provided $q \neq 0$ on the free surface. For cohesionless materials which satisfy the Coulomb-Mohr yield condition (7), we have $q = 0$ on the free surface, and in this case the analysis is no longer appropriate.

As a model problem, we have examined a problem which occurs when flowing material in hoppers is blocked by the formation of stable, cohesive arches. Historically, most of the research on this subject has been directed towards calculating arching criteria, so that hoppers can be designed with the prevention of arching in mind. In the current paper we are unable to derive any results for such arching criteria, but instead consider two limiting cases (granular materials with small cohesion $0 < c \ll 1$ and highly friction materials $\beta = 1$), where it is possible to use existing solutions for quasi-steady flow in a hopper to determine the shape of the limiting free boundaries.

For the case of small cohesion $0 < c \ll 1$, the limiting free boundary is $O(c)$ away from the apex. In this case the leading order solution for the stresses is not surprisingly the radial solution for cohesionless quasi-steady flow in infinite hopper. The highly frictional limit $\beta = 1$ is particularly interesting, since we are able to present exact solutions to the governing equations which satisfy all the boundary conditions. In this limit, a granular material is able to form stable, self-supporting structures which are independent of the cohesion.

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